Proetale cohomology of affine spaces (§3 of "Espace de Banach-Colmez et Faisceaux Cohérents sur La Courbe de Fargues-Fontaine")

Notations : H^i ⇝ proetale cohomology
H^i_{et} ⇝ etale cohomology

$\mathbb{C} := \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ $\qquad$ $\mathbb{A}^n_{\mathbb{C}}$ = analytification of n-dim affine space / $\mathbb{C}$ $= \bigcup_{m>0} D_m$

D_m = open ball of radius m

$\overline{B}_n$ = closed ball of radius $p^{-\frac{1}{n}}$

B_n = open "

$D = D_1 = \bigcup_{n>0} \overline{B}_n = \bigcup B_n$

$\nu' : \mathbb{A}^n_{\mathbb{C}, \text{proet}} \to \mathbb{A}^n_{\mathbb{C}, \text{et}}$

Main result : 1) $H^i(\mathbb{A}^n_{\mathbb{C}}, \mathbb{G}_a) = \Omega^i(\mathbb{A}^n_{\mathbb{C}})$ $\qquad i \geq 0,\ n \geq 1$

2) $H^i(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Q}_p) = \ker(d_i) = \operatorname{im}(d_{i-1}) \subset \Omega^i(\mathbb{A}^n_{\mathbb{C}})$

$$\Omega^\bullet(\mathbb{A}^n_{\mathbb{C}}) : \mathcal{O}(\mathbb{A}^n_{\mathbb{C}}) \xrightarrow{d_0} \Omega^1(\mathbb{A}^n_{\mathbb{C}}) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(\mathbb{A}^n_{\mathbb{C}})$$

preliminaries :

prop 3.5 $\quad$ · I : ordered filtering set with countable cofinal part

· $\{A_i\}_{i \in I}$: proj system of abelian groups

1) If $\{A_i\}_{i\in I}$ satisfies Mittag-Leffler, then

$$R^1 \varprojlim A_i = 0 \qquad (*)$$

2) If A_i are complete metric spaces (compatible with gp structures)

s.t f_{ij} are uniformly continuous & for every $i \in I$, $\exists$ $j \geq i$ s.t for every $k \geq j$, $f_{ik}(A_k)$ is dense in A_i

Then $(*)$ holds as well.

Prop 3.6
- I as above
- X analytic adic space
- $\{\mathcal{F}_i\}_{i \in I}$ proj system of abelian sheaves on X_{proet} s.t f_{ij} are all surjective

Then $R^k \varprojlim \mathcal{F}_i = 0$, $\forall$ $k > 0$

Prop 3.7 X analytic adic space $/\mathbb{C}$. $\mathcal{F}$ abelian sheaf on X_{et}
$\nu : \hat{X}_{proet} \to \hat{X}_{et}$. Then $H^i(X, \nu^* \mathcal{F}) = H^i_{et}(X, \mathcal{F})$

Berkovich's IHES paper: a) $H^i_{et}(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Z}/p^k) = 0$ $\forall$ $i > 0$, $k \geq 1$
$\rightsquigarrow H^i(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Z}/p^k) = 0 \rightsquigarrow H^i(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Z}_p) = 0$ $i > 0$
(ok b/c $\mathbb{A}^n_{\mathbb{C}}$ not qcqs)

b) $H^i_{et}(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Z}/p^k) = 0$, $i > 2$

3) $H^2_{et}(\bar{B}_n, \mathbb{G}_m) = 0$

proof of Main result

proof of 1) : prop 3.4 $R^i \nu'_* \mathbb{G}_a = \Omega^i_{\mathbb{A}^n_{\mathbb{C}}}$

proof : $\mathbb{G}_a = \hat{\mathcal{O}}$ on $\mathbb{A}^n_{\mathbb{C}, proet}$. Then this is proved by Scholze in his CDM paper.

$= 0$ if $j > 0$

$\rightsquigarrow \quad H^j_{et}(\mathbb{A}^n_{\mathbb{C}}, \Omega^i) \Rightarrow H^{i+j}(\mathbb{A}^n_{\mathbb{C}}, G_n)$

$\Rightarrow 1)$

proof of 2) :

Two approaches : · special one : for $n=1$

· general one : for all n.

<u>special one</u> :

prop 3.8 a) $H^i(D, \mathbb{Q}_p) = 0$ if $i > 1$

b) $H^1(D, \mathbb{Q}_p) \simeq \mathcal{O}(D)_0 = \{ f \in \mathcal{O}(D) \mid f(0) = 0 \}$

Cor 3.10 a) $H^i(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Q}_p) = 0 \quad i > 1$

b) $H^0(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Q}_p) = \mathbb{Q}_p$

c) $H^1(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Q}_p) \simeq \mathcal{O}(\mathbb{A}^1_{\mathbb{C}})_0 \simeq \mathrm{Im}(d_0)$

<u>proof</u> $\mathbb{A}^1_{\mathbb{C}} = \bigcup_{m>0} D_m \rightsquigarrow 0 \to R^1\varprojlim H^{i-1}(D_m, \mathbb{Q}_p) \to H^i(\mathbb{A}^n_{\mathbb{C}}, \mathbb{Q}_p)$

$\to \varprojlim H^i(D_m, \mathbb{Q}_p) \to 0$

$i > 2$ ✓

$i = 2$ ✓ (prop 3.5 + density)

$i = 1$ ✓ (prop 3.5 + MF)

<u>proof of prop 3.8</u>

Berkovich : $H^i_{et}(\bar{B}_n, \mathbb{Z}/p^k) = 0 \quad i > 2$

Step 1 : On $\bar{B}_n$ Kummer theory $\rightsquigarrow$

$$0 \to \mathcal{O}(\overline{B}_n)^\times / (\mathcal{O}(\overline{B}_n)^\times)^{p^k} \to H^1_{et}(\overline{B}_n, \mathbb{Z}/p^k) \to \underbrace{H^1_{et}(\overline{B}_n, \mathbb{G}_m)}_{=0} \text{ (Berkovich;}$$

$$\to H^1_{et}(\overline{B}_n, \mathbb{G}_m) \to H^2_{et}(\overline{B}_n, \mathbb{Z}/p^k) \to H^2_{et}(\overline{B}_n, \mathbb{G}_m) \to \cdots$$

$$\underset{0}{\|} \text{ Lut}\ldots$$

$$\Rightarrow H^2_{et}(\overline{B}_n, \mathbb{Z}/p^k) = 0 \qquad . \qquad H^1_{et}(\overline{B}_n, \mathbb{Z}/p^k) \simeq \mathcal{O}(\overline{B}_n)^\times / (\mathcal{O}(\overline{B}_n)^\times)^{p^k}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$H^2_{et}(\overline{B}_n, \mathbb{Q}_p) = 0 \qquad H^1_{et}(\overline{B}_n, \mathbb{Z}_p) = \varprojlim \mathcal{O}(\overline{B}_n)^\times / (\mathcal{O}(\overline{B}_n)^\times)^{p^k}$$

Note that $\mathcal{O}(\overline{B}_n)^\times = \mathbb{C}^\times \cdot \{ f = 1 + \sum_{i \geq 1} a_i X^i \, , \underbrace{|a_i| p^{-i/n} < 1, \forall i}_{\Uparrow \text{ Newton-Polygon}} \}$

$$\mathcal{O}(\overline{B}_n)^\times / (\mathcal{O}(\overline{B}_n)^\times)^{p^k} = M_n / M_n^{p^k} \qquad \overset{\|}{M_n}$$

For $f \in M_n$ $\quad \log f \big|_{\overline{B}_{n-1}} \in \mathcal{O}(\overline{B}_{n-1})_0^{\leq r}$ $\quad$ for some fixed r

$$f \in M_n^{p^k} \Rightarrow \log f \big|_{\overline{B}_{n-}} \in p^k \mathcal{O}(\overline{B}_{n-1})^{\leq r}$$

$$\rightsquigarrow \Phi_n : H^1_{et}(\overline{B}_n, \mathbb{Q}_p) = \left(\varprojlim M_n / M_n^{p^k} \right) [\tfrac{1}{p}] \xrightarrow{\text{restriction}} \mathcal{O}(\overline{B}_{n-1})_0$$

$$0 \to R^1\varprojlim H^0(\overline{B}_n, \mathbb{Z}/p^k) \to H^1(\overline{B}_n, \mathbb{Z}_p) \to \varprojlim H^1(\overline{B}_n, \mathbb{Z}/p^k) \to 0$$

$$\| \qquad\qquad$$

$$H^1_{et}(\overline{B}_n, \mathbb{Z}/p^k) \to 0$$

$$\Rightarrow H^1(\overline{B}_n, \mathbb{Z}_p) = H^1_{et}(\overline{B}_n, \mathbb{Z}_p)$$

$$0 \to R^1\varprojlim \overset{/\!/^0}{H^0(\overline{B}_n, \mathbb{Q}_p)} \to H^1(D, \mathbb{Q}_p) \to \varprojlim H^1(\overline{B}_n, \mathbb{Q}_p) \to 0$$

$$\rightsquigarrow \Phi : H^1(D, \mathbb{Q}_p) \to \varprojlim \mathcal{O}(\overline{B}_{n-1})_0 = \mathcal{O}(D)_0$$

Check: this is an isom.

$$\| \ \varprojlim H^1(\overline{B}_n, \mathbb{Q}_p)$$

For $i \geq 2$ $\quad 0 \to R^1\varprojlim H^{i-1}(B_n, \mathbb{Q}_p) \to H^i(D, \mathbb{Q}_p) \to \varprojlim H^i(B_n, \mathbb{Q}_p) \to 0$

$i > 2 \quad = 0$

$i = 2 \qquad R^1\varprojlim H^1(B_n, \mathbb{Q}_p) = 0 \qquad$ (prop 3.5 + dense image)

$\mathcal{O}(B_{n+1})_0 \to \mathcal{O}(B_n)_0$ dense image